In contrast:

Teachers face a dilemma when they try to move children to do school work that is not intrinsically interesting. Children must be induced to undertake the work either by promise of reward or threat of punishment, and in neither case do they focus on the material to be learnt. In this sense the work is construed as a bad thing, an obstacle blocking the way to reward of a reason for punishment. Kurt Lewin explores this dilemma in "The Psychological Situation of Reward and Punishment" (*A Dynamic Theory of Personality: Selected Papers of Kurt Lewin*, McGraw-Hill, 1935). The ideas of Piaget and Lewin have led me to state the central problem of education thus: How can we instruct while respecting the self-constructive character of mind? (Lawler, 1982, p. 138.)

OVERVIEW

The rest of this book is about human intelligence and the learning of mathematics. Part A, Chapters 2 to 7, are as they appeared in the original edition published by Penguin in 1971. Since these chapters first appeared, there has been an increasing amount of valuable work in this field, much of it inspired by the pioneering work of Piaget. If I were starting now from the beginning, there would of course be many references to this work. The result would be quite a different book, more in the nature of a survey; and there are already books in print which do this different job well. But because there is nothing in the original chapters about which I now think differently, it has seemed better not to risk changing what is still being well received in seven languages, but to add a sequel. This forms Part B.

The order in which the chapters now appear is one good order in which to read them. However, for those entirely new to the subject, Chapter 12 provides a good introduction. Since it first appeared in the journal *Mathematics Teaching*, this has been read by more people than anything else I have written, and it fits well with the intuitions of many. The full theoretical underpinning for these ideas will be understood later, when returning to this chapter in its numerical sequence. Another good order would be to read chapter 8 first, and then the earlier chapters. This plan will allow the reader to see where I was going somewhat better than, at the time, I did myself.

Skemp, Richard R. The Psychology of Learning mathematics. Expanded American ed. Hillsdale, NJ: Lawrence Erlbaum, 1987

2

The Formation of Mathematical Concepts

In this chapter we shall examine what we mean by concepts, and how we form, use and communicate these. Then, in Chapter 3, we shall consider how concepts fit together to form conceptual structures, called schemas, and examine some of the results which follow from the organization of our knowledge into these structures.

ABSTRACTING AND CLASSIFYING

Though the term 'concept' is widely used, it is not easy to define. Nor, for reasons which will appear later, is a direct definition the best way to convey its meaning. So I shall approach it from several directions, and with a variety of examples. Since mathematical concepts are among the most abstract, we shall reach these last.

First, two pre-verbal examples. A baby boy aged twelve months, having finished sucking his bottle, crawled across the floor of the living room to where two empty wine bottles were standing and stood his own empty feeding bottle neatly alongside them. A two-year-old boy, seeing a baby on the floor, reacted to it as he usually did to dogs, patting it on the head and stroking its back. (He had seen plenty of dogs, but had never before seen another baby crawling.)

In both these cases the behaviour of the children concerned implies: one, some kind of classification of their previous experience; two, the fitting of their present experience into one of these classes.

We all behave like this all the time; it is thus that we bring to bear our past experiences on the present situation. The activity is so continuous and automatic that it requires some slightly unexpected outcome thereof, such as the above, to call it to our attention.

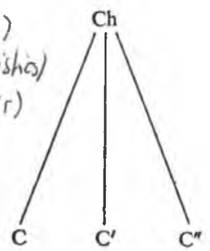
At a lower level we classify every time we recognize an object as one which we have seen before. On no two occasions are the incoming sense-data likely to be exactly the same, since we see objects at different distances and angles, and also in varying lights. From these varying inputs we abstract certain *in*variant properties, and these properties persist in memory longer than the memory of any particular presentation of the object. In the diagram, C_1, C_2 . . . represent suc-

Recognizing characteristics held in common

cessive past experiences of the same object, say, a particular chair. From these we abstract certain common properties, represented in the diagram by C. Once this abstraction is formed, any further experience, C_n, evokes C, and the chair is *recognized:* that is, the new experience is classified with C_1, C_2, etc.; C_n and C are now experienced together; and from their combination we experience both the *similarity* (C) of C_n to our previous experiences of seeing this chair and also the particular distance, angle, etc., on this occasion (C_n).

We progress rapidly to further abstractions. From particular chairs, C, C', C'', we abstract further invariant properties, by which we recognize *Ch* (a new object seen for the first time, say, in a shop window) as a member of this class. It is the second-order abstraction (from the set of abstractions C, C', . . .) to which we give the name 'chair.' The invariant properties which characterize it are already

The invariant (common) properties (or characteristics) inherit from Ch (chair) to C, C', C'' (each particular chair)



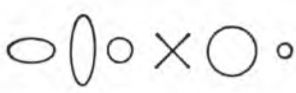
becoming more functional and less perceptual—that is, less attached to the physical properties of a chair. One I saw recently was of basket-work, eggshaped and hung from a single rope. It bore little or no physical resemblance to any chair which I had ever seen—but I recognized it at once as a chair, and a most desirable one too!

From the abstraction *chair,* together with other abstractions such as *table, carpet, bureau,* a further abstraction, *furniture,* can be made, and so on. These classifications are by no means fixed. Particularly by the young, chairs are also classified as things to stand on, gymnastic apparatus and framework of a play house. Tables are sometimes used as chairs. This flexibility of classification, according to the needs of the moment, is clearly an aid to adaptability.

Naming an object classifies it. This can be an advantage or a disadvantage. A very important kind of classification is by function, and once an object is thus classified, we know how to behave in relation to it. 'Whatever is this?' 'It's a gadget for pulling off Wellington boots.' But once it is classified in a particular way, we are less open to other classifications. Most of us classify cars as vehicles, time-savers and perhaps status symbols, and use them in accordance with these functions. Fewer also see them as potentially lethal objects, and our behaviour therefore takes less account than it should of this classification.

It may be useful to bring together some of the terms used so far. *Abstracting* is an activity by which we become aware of similarities (in the everyday sense) among our experiences. *Classifying* means collecting together our experiences on the basis of these similarities. An *abstraction* is some kind of lasting mental change, the result of abstracting, which enables us to recognize new experiences as having the similarities of an already formed class. Briefly, it is something learnt which enables us to classify; it is the defining property of a class. To distinguish between abstracting as an activity and an abstraction as its end-product, we shall hereafter call the latter a *concept.*

A concept therefore requires for its formation a number of experiences which have something in common. Once the concept is formed, we may (retrospectively and prospectively) talk about *examples* of the concept. Everyday concepts come from everyday experience, and the examples which lead to their formation occur randomly, spaced in time. More frequently encountered objects are, in general, conceptualized more rapidly; but in many other factors are at work which make this statement an oversimplification. One of these is *contrast.* In the diagram on the right the single X stands out perceptually from the five variously shaped Os. Objects which thus stand out from their surroundings are more likely to be remembered and



their similarities are more likely to be abstracted across intervals of space and time.

The diagram also illustrates the function of non-examples in determining a class. The X, by its difference from all the other shapes, makes the similarity between them more noticeable. The essential characteristics of a *chair* are clarified by pointing to, say, a stool, a settee, a bed and a garden seat, and saying

'These are not chairs.' This is specially useful in fixing the borderline of a class—we use objects which might be examples, but aren't.

NAMING

We have just used *naming* again. Language is, in humans, so closely linked with concepts and concept-formation that we cannot for long keep naming out of our discussion. Indeed, many people find it difficult to separate a concept from its name, as is shown by the following charming illustration provided by Vygotsky (1962). Children were told that in a game a dog would be called 'cow.' The following is a typical sequence of questions and answers. 'Does a cow have horns?' 'Yes.' 'But don't you remember that the cow is really a dog? Come now, does a dog have horns?' 'Sure, if it is a cow, if it's called cow, it has horns. That kind of dog has got to have little horns.' Vygotsky also quotes a story about a peasant who, after listening to two students of astronomy talking about the stars, said that he could understand that with the help of instruments people could measure the distance from the earth to the stars and find their positions and motion. What puzzled him was how in the devil they found out the *names* of the stars!

The distinction between a concept and its name is an essential one for our present discussion. A concept is an idea; the name of a concept is a sound, or a mark on paper, associated with it. This association can be formed after the concept has been formed ('What is this called?') or in the process of forming it. If the same name is heard or seen each time, an example of a concept is encountered, by the time a concept is formed, the name has become so closely associated with it that it is not only by children that it is mistaken for the concept itself. In particular, numbers (which are mathematical concepts) and numerals (the names we use for numbers) are widely confused. This point is discussed further in Chapter 4.

Being associated with a concept, the use of a name in connection with an object helps us to classify it, that is, to recognize it as belonging to an existing class. 'What's this?' 'A new kind of bottle opener which works by compressed air.' Now we have classified it, which we were unable to do by its perceptual properties alone; so we know what to do with it. This classification was done by bringing the concept of a bottle opener to consciousness at the same time as the new experience.

Naming can also play a useful, sometimes an essential, part in the formation of new concepts. Hearing the same name in connection with different experiences predisposes us to collect them together in our minds and also increases our chance of abstracting their intrinsic similarities (as distinct from the extrinsic one of being called by the same name). Experiment has also shown that associating different names with classes which are only slightly different in their characteristics helps to classify later examples correctly, even if the later examples are not named. The names help to separate the classes themselves.

THE COMMUNICATION OF CONCEPTS

We can see that language can be used to speed up the formation of a concept by helping to collect and separate contributory examples and non-examples. Can it be used to short-circuit the process altogether by simply defining a concept verbally? Particularly in mathematics, this is often attempted, so let us examine the idea of a definition, as usual with the help of examples. To begin with, let us choose a simple and well-known concept, say, *red,* and imagine that we are asked the meaning of this word by a man, blind from birth, who has been given sight by a corneal graft. The meaning of a word is the concept associated with that word, so our task is now to enable the person to form the concept *red* (which he does not have when we begin) and associate it with the word 'red.'

There are two ways in which we might do this. Being scientifically inclined, and perhaps interested in colour photography, we could give a definition: 'Red is the colour we experience from light of wavelength in the region of 0·6 microns.' Would he now have the concept red? Of course not. Such a definition would be useless *to him,* though not necessarily for other purposes. Intuitively, in such a case, we would point to various objects and say 'This is a red diary, this is a red tie, this is a red jumper . . .' In this way we would arrange for him to have, close together in time, a collection of experiences from which we hope he will abstract the common property—red. Naming is here used as an auxiliary, in the way already described. The same process of abstraction could take place in silence, but it would probably be slower and the name 'red' would not become attached.

If he now asks a different question, 'What does "colour" mean?', we can no longer collect together examples for him by pointing, for the examples we want are *red, blue, green, yellow . . .*, and these are themselves concepts. If, and only if, he already has these concepts in his own mind—their presence in our mind is not enough—then, by collecting together the words for them, we can arrange for him to collect together the concepts themselves, and thus make possible, though not guarantee, the process of abstraction. Naming (or some other symbolization) now becomes an essential factor of the process of abstraction and not just a useful help.

This leads us to an important distinction between two kinds of concept. Those which are derived from our sensory and motor experiences of the outside world, such as *red, motor car, heavy, hot, sweet,* will be called *primary concepts;* those which are abstracted from other concepts will be called *secondary concepts.* If concept A is an example of concept B, then we shall say that B is of a higher order than A. Clearly, if A is an example of B, and B of C, then C is also of higher order than both B and A. 'Of higher order than' means 'abstracted from' (directly or indirectly). So 'more abstract' means 'more removed from experience of the outside world,' which fits in with the everyday meaning of the word 'abstract.' This comparison can only be made between concepts in the same hierarchy. Although we might consider that *sonata form* is a more abstract (higher order) concept than *colour,* we cannot properly compare the two.

These related ideas, of order between concepts and a conceptual hierarchy, enable us to see more clearly why, for the person we are thinking of, the definition of red was an inadequate mode of communication: it presupposed concepts such as *colour, light,* which could only be formed if concepts such as *red, blue, green* . . . had already been formed. In general, *concepts of a higher order than those which people already have cannot be communicated to them by a definition* but only by collecting together, for them to experience, suitable examples.

Of what use, then, if any, is a definition?

Two uses can be seen at once. If it were necessary (for example, for a photographic colour filter) to specify exactly within what limits we would still call a colour red, then the above definition would enable us to say where red starts and finishes. And having gone further in the process of abstraction, that is, in the formation of larger classes based on similarities, a definition enables us to retrace our steps. By stating all those (and only those) classes to which our particular concept belongs, we are left with just one possible concept—the one we are defining. In the process we have shown how it relates to the other concepts in its hierarchy. Definitions can thus be seen as a way of adding precision to the boundaries of a concept, once formed, and of stating explicitly its relation to other concepts.

New concepts of a lower order can also be communicated for the first time by this means. For example, if our formerly blind subject asked 'What colour is magenta?' and we could not find a sufficiency of magenta objects to show him, we could say 'It is a colour, between red and blue, rather more blue than red.' Provided that he already had the concepts of blue and red, he could then form at least a beginning of the concept of magenta without ever having seen this colour.

Since most of the new concepts we need in everyday life are of a fairly low order, we usually have available suitable higher-order concepts for the new concepts to be easily communicable by definition, often followed by an example or two, which then serve a different purpose—that of illustration. 'What is a stool?' 'It's a seat without a back for one person' is quite a good definition, but even so a few examples will define the concept in such a way as to exclude hassocks, pouffes and garden swings far more successfully than further elaboration of the definition.

In mathematics, however, not only are the concepts far more abstract than those of everyday life, but the direction of learning is for the most part in the direction of still greater abstraction. The communication of mathematical concepts is therefore much more difficult, on the part of both communicator and receiver. This problem will be taken up again shortly, after certain other general topics have been explored.

CONCEPTS AS A CULTURAL HERITAGE

Low-order concepts can be formed, and used, without the use of language.

The criterion for *having* a concept is not being able to say its name but behaving in a way indicative of classifying new data according to the similarities which go to form this concept. Animals behave in ways from which one may reasonably infer that they form simple concepts. A rat, trained to choose a door coloured mid-grey in preference to a light grey, will if now presented with doors of mid-grey and dark grey go to the dark grey. It processes the data in terms of 'darker than.'

The most obvious discontinuity between human beings and other animals is in the former's use of language. What this implies is less obvious. If we choose a word at random it will almost always be found that the concept which the word names—the meaning of the word—is not a specific object or experience but a class. (Proper nouns are a partial exception.)

Now, there are two ways of evoking a concept, that is, of causing it to start functioning. One is by encountering an example of the concept. The concept then comes into action as our way of classifying this example, and our subjective experience is that of *recognition*. The other is by hearing, reading or otherwise making conscious the name, or other symbol, for the concept. Animals can do the first; only human beings can do the second. And the reason for this lies not in superior vocal apparatus, but *in the ability to isolate concepts from any of the examples which give rise to them*. Only by being detachable from the sensory experiences from which they originated can concepts be collected together as examples from which new concepts of greater abstraction can be formed.

We would expect this detachability to be related to abstracting ability, for the stronger the mental organization based not on direct sense-data but on similiarities between them, the greater we would expect its ability to function as an independent entity. This view is supported by evidence from several sources. Children of very low intelligence do not learn to talk, in spite of adequate vocal apparatus. Chimpanzees, the closest of our surviving ancestors, can learn to sit at a table and drink from a cup, but not to talk. Human beings are the most intelligent and the most adaptable of all species. They are also the only species who can talk.

Our ability to make concepts independent of the experiences which gave rise to them and to manipulate them by the use of language is the very core of human superiority over other species. This is the first step towards the realization of the potential which this greater intelligence gives. Intelligence makes speech possible, but speech (which has to be learnt) is essential for the formation and use of the higher-order concepts which, collectively, form our scientific and cultural heritage.

A concept is a way of processing data which enables the user to bring past experience usefully to bear on the present situation. Without language each individual has to form his own concept direct from the environment. Without language, these primary concepts cannot be brought together to form concepts of higher order. By language, however, the first process can be speeded up and the second is made possible. Moreover, the concepts of the past, painstakingly abstracted and slowly accumulated by successive generations, become available to help each new individual form his own conceptual system.

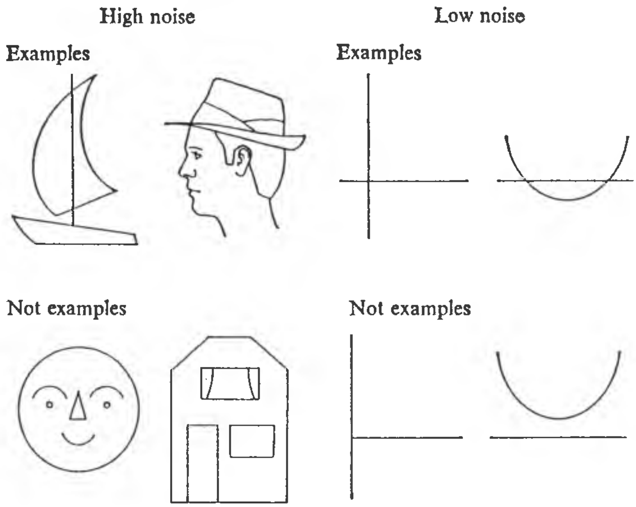
The actual construction of a conceptual system is something which individuals have to do for themselves. But the process can be enormously speeded up if, so to speak, the materials are to hand. It is like the difference between building a boat from a kit of wood already sawn to shape and having to start by walking to the forest, felling the trees, dragging them home, making planks—having first mined some iron ore and smelted it to make an axe and a saw!

What is more, the work of geniuses can be made available to everyone. Concepts like that of gravitation, the result of years of study by one of the greatest intelligences the world has known, become available to all scientists who follow. The first person to form a new concept of this order has to abstract it relatively unaided. Thereafter, language can be used to direct the thoughts of those who follow so that they can make the same discovery in less time and with less intelligence. Yet even Newton (1642–1727) was by no means altogether unaided. He said, with modesty, 'If I have seen a littler farther than others, it is because I have stood on the shoulders of giants.' The conceptual structures of earlier mathematicians and scientists were available to him.

In this context, the generalized idea of *noise* is useful. By this is meant data which is irrelevant to a particular communication, so that what is noise in one context may not be so in another. (For example, if we are listening to music when the telephone rings, the sound of the bell conveys *information* that someone is calling us, but is *noise* relative to the music.) The greater the noise, the harder it is to form the concept. Before reading on, please put your hand over the diagrams which are on the right-hand side on the next page. Try to form the concept from the high-noise examples and non-examples. Now remove your hand and try to form the concept from the low-noise examples of the same concept.

From the right-hand examples it is much easier to see that the concept is *having intersecting lines*. The extra noise in the left-hand examples comes partly from the additional lines, but largely from the fact that each looks like something.

An attribute of high intelligence is the ability fo form concepts under conditions of great noise. But once we have a concept, we can see examples of it where previously we could not.



THE POWER OF CONCEPTUAL THINKING

Conceptual thinking confers on users great power to adapt their behaviour to the environment, and to shape their environment to suit their own requirements. This results partly from the detachment of the concepts from both present sense-data and behaviour, and their manipulation independently of these. We take this so much for granted that we hardly realize the enormous advantage of *not* having to do something in order to discover whether it is the best thing to do! But, of course, all major activities, from setting up in business to building an aircraft, are put together in thought before they are constructed in fact.

The power of concepts also comes from their ability to combine and relate many different experiences and classes of experience. The more abstract the concepts, the greater their power to do this. The person who says 'Don't worry me with theory—just give me the facts' is speaking foolishly. A set of facts can be used only in the circumstances to which they belong, whereas an appropriate theory enables us to explain, predict and control a great number of particular events in the classes to which it relates.

A further contribution to the power of conceptual thinking is related to the shortness of our span of attention. Our short-term memory can only store a limited number of words or other symbols. Clearly the higher the order of the concepts which these symbols represent, the greater the stored experience they bring to bear. Mathematics is the most abstract, and so the most powerful, of all theoretical systems. It is therefore potentially the most useful; scientists in particular, but also economists and navigators, businessmen and communications engineers, find it an indispensable 'tool' (data-processing system) for their work.

Its usefulness is, however, only potential, and many who work wearily at trying to learn it throughout their schooldays derive little benefit, and no enjoyment. This is almost certainly because they are not really learning mathematics at all. The latter is an interesting and enjoyable process, though many will find this hard to believe. What is inflicted on all too many children and older students is the manipulation of symbols with little or no meaning attached, according to a number of rote-memorized rules. This is not only boring (because meaningless); it is very much harder, because unconnected rules are much harder to remember than an integrated conceptual structure. The latter point will be taken up in the next chapter. Here we shall concentrate on the communication of mathematical concepts.

THE LEARNING OF MATHEMATICAL CONCEPTS

Much of our everyday knowledge is learnt directly from our environment, and the concepts involved are not very abstract. The particular problem (but also the power) of mathematics lies in its great abstractness and generality, achieved by successive generations of particularly intelligent individuals each of whom has been abstracting from, or generalizing, concepts of earlier generations. The present-day learner has to process not raw data but the data-processing systems of existing mathematics. This is not only an immeasurable advantage, in that an able student can acquire in years ideas which took centuries of past effort to develop; it also exposes the learner to a particular hazard. Mathematics cannot be learnt directly from the everyday environment, but only indirectly from other mathematicians, in conjunction with one's own reflective intelligence. At best, this makes one largely dependent on teachers (including all who write mathematical textbooks); at worst, it exposes one to the possibility of acquiring a lifelong fear and dislike of mathematics.

Though the first principles of the learning of mathematics are straightforward, it is the communicator of mathematical ideas, and not the recipient, who most needs to know them. And though they are simple enough in themselves, their mathematical applications involve much hard thinking. The first of these principles was stated earlier in the chapter:

(1) Concepts of a higher order than those which people already have cannot be communicated to them by a definition, but only by arranging for them to encounter a suitable collection of examples.

The second follows directly from it:

(2) Since in mathematics these examples are almost invariably other concepts, it must first be ensured that these are already formed in the mind of the learner.

The first of these principles is broken by the vast majority of textbooks, past and present. Nearly everywhere we see new topics introduced not by examples but by definitions, definitions of the most admirable brevity and exactitude for the teacher (who already has the concepts to which they refer) but unintelligible to the student. For reasons which will be apparent, examples cannot be quoted here, but readers are invited to verify this statement for themselves. It is also a useful exercise to look at some definitions of ideas new to oneself in books about mathematics beyond the stage which one has reached. This enables one to experience at first hand the bafflement of the younger learner.

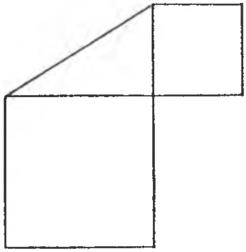
Good teachers intuitively help out a definition with examples. To choose a suitable collection is, however, harder than it sounds. The examples must have in common the properties which form the concept but no others. To put it differently, they must be alike in the ways which are to be abstracted, and otherwise different enough for the properties irrelevant to this particular concept to cancel out or, more accurately, fail to summate. Remembering that these irrelevant properties may be regarded as noise, we may say that some noise is necessary to concept formation. In the earlier stages, low noise—clear embodiment of the concept, with little distracting detail—is desirable; but as the concept becomes more strongly established, increasing noise teaches the recipient to abstract the conceptual properties from more difficult examples and so reduces dependence on the teacher.

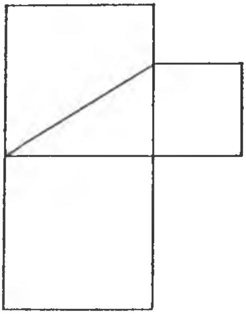
Composing a suitable collection thus requires both inventiveness and a very clear awareness of the concept to be communicated. Now, it is possible to have, and use, a concept at an intuitive level without being consciously aware of it. This applies particularly to some of the most basic and frequently used ideas: partly because the more automatic any activity, the less we think about it; partly because the most fundamental ideas of mathematics are acquired at an early age, when we have not the ability to analyse them; and partly because some of these fundamental ideas are also among the most subtle. But it is easy to slip up even when these factors do not apply.

Some children were learning the theorem of Pythagoras (*c.* sixth century BC). They had copied a right-angled triangle from the blackboard—figure a—and were told to make a square on each side. This they did easily enough for the two shorter sides—figure b; but they were nearly all in difficulty when they tried to draw the square on the hypotenuse. Many of them drew something like figure c. From this, I inferred that the squares from which they had formed their concepts had all been 'square' to the paper and had included no obliquely placed examples. All too easily done!

The second of the two principles, that the necessary lower-order concepts must be present before the next stage of abstraction is possible, seems even more straightforward. To put this into effect, however, means that before we try to communicate a new concept, we have to find out what are its contributory concepts; and for each of these, we have to find out *its* contributory concepts,


a


b


c

and so on, until we reach either primary concepts or experience which we can assume. When this has been done, a suitable plan can then be made which will present to the learner a possible, and not an impossible, task.

This conceptual analysis involves much more work than just giving a definition. If done, it leads to some surprising results. Ideas which not long ago were first taught in university courses are now seem to be so fundamental that they are being introduced in the primary school: for example, sets, one-to-one correspondence. Other topics still regarded as elementary are found on analysis to involve ideas which even those teaching the topic have for the most part never heard of. In this category I include the manipulation of fractional numbers.

There are two other consequences of the second principle. The first is that in the building up of the structure of successive abstractions, if a particular level is imperfectly understood, everything from then on is in peril. This dependency is probably greater in mathematics than in any other subject. One can understand the geography of Africa even if one has missed that of Europe; one can understand the history of the nineteenth century even if one has missed that of the eighteenth; in physics one can understand 'heat and light' even if one has missed 'sound.' But to understand algebra without ever having really understood arithmetic is an impossibility, for much of the algebra we learn at school is generalized arithmetic. Since many pupils learn to do the manipulations of arithmetic with a very imperfect understanding of the underlying principles, it is small wonder that mathematics remain a closed book to them. Even those who get off to a good start may, through absence, inattention, failure to keep up with the pace of the class or other reasons, fail to form the concepts of some particular stage. In that case, all subsequent concepts dependent on these may never be understood, and pupils become steadily more out of their depth. In the latter case, however, the situation may not be so irremediable, if the learning situation is one which makes back-tracking possible: for example, if the text in use provides a genuine explanation and is not just a collection of exercises. Success will then depend partly on the confidence of the learners in their own powers of comprehension.

The other consequences (of the second principle) is that the contributory concepts needed for each new stage of abstraction must be *available*. It is not sufficient for them to have been learnt at some time in the past; they must be accessible when needed. This is partly a matter, again, of having facilities available for back-tracking. Appropriate revision, planned by a teacher, will be specially useful for beginners, but more advanced students should be taking a more active part in the direction of their own studies, and, for these, returning to take another look at earlier work will be more effective if it is directed by a felt need rather than by an outside instruction. To put it differently, an answer has more meaning to someone who has first asked a question.

LEARNING AND TEACHING

In learning mathematics, although we have to create all the concepts anew in our own minds, we are only able to do this by using the concepts arrived at by past mathematicians. There is too much for even a genius to do in a lifetime.

This makes the learning of mathematics, especially in its early stages and for the average student, very dependent on good teaching. Now, to know mathematics is one thing and to be able to teach it—to communicate it to those at a lower conceptual level—is quite another; and I believe that it is the latter which is most lacking at the moment. As a result, many people acquire at school a lifelong dislike, even fear, of mathematics.

It is good that widespread efforts have been and are still being made to remedy this, for example, by the introduction of new syllabi, more attractive presentation, television series and other means. But the small success of these efforts, after twenty years or more, supports the view already put forward in the introduction, namely; that these efforts will be of little value until they are combined with greater awareness of the mental processes involved in the learning of mathematics.

3

The Idea of a Schema

Though in the previous chapter our attention was centred on the formation of single concepts, each of these by its very nature is embedded in a structure of other concepts. Each (except primary concepts) is derived from other concepts and contributes to the formation of yet others, so it is part of a hierarchy. But at each level alternative classifications are possible, leading to different hierarchies. A car can be classed as a vehicle (with buses, trains, aircraft), as a status symbol (with a title, a good address, a mink coat), as a source of inland revenue (with tobacco, drink, and dog licenses), as an export (with gramophone records, Scotch whisky, Harris tweed), etc. What is more, the class concepts on which we have been concentrating so far are by no means the only kind. Given a collection not of single objects but of *pairs* of objects we may become aware of something in common between the pairs. For example:

puppy, dog; kitten, cat; chicken, hen.

Here we see that each of these pairs can be connected by the idea '. . . is a young . . .' Another example:

Bristol, England; Hull, England; Rotterdam, Holland.

In this, each pair can be connected by the idea '. . . is a port of . . .' These two connecting ideas are themselves examples of a new idea called a *relation*.

A mathematical relation may be seen in the following collection of pairs.

(6, 5), (2, 1), (9, 8), (32, 31) . . .

We can call this relation 'is one more than' or 'is the successor of.' Another mathematical example:

$$(\tfrac{1}{2}, \tfrac{2}{4}), \quad (\tfrac{1}{3}, \tfrac{2}{6}), \quad (\tfrac{1}{4}, \tfrac{2}{8}) \ldots$$

This relation is called 'is equivalent to'. The fractions in each pair, though not identical, represent the same number. Notice (1) that in mathematics it is usual to enclose the pairs in a given relation in parentheses, as above; (2) that the order within the pairs usually matters. These:

$$(5, 6), \quad (1, 2), \quad (8, 9), \quad (31, 32)$$

are in a different relation to these:

$$(6, 5), \quad (2, 1), \quad (9, 8), \quad (32, 31)$$

We can even start to classify these relations. Those mathematical relations given as examples in the last paragraph were chosen to exemplify two particular kinds: order relations and equivalence relations. Other order relations are: is greater than, is the ancestor of, happened after. Other equivalence relations are: is the same size as, is the sibling of, is the same colour as. Both order relations and equivalence relations have important general properties. So we have not only a hierarchical structure of class concepts but another structure of individual relations, and classes of relations, which forms cross-linkages within the first structure.

Another source of cross-linkages arises from our ability to 'turn one idea into another' by doing something to it.

Example:	good→bad	hot→cold	high→low
Another example:	good→best	bad→worst	high→highest

This 'something which we can do to an idea' is called a *transformation*, or more generally a *function*. There are many different kinds of transformation, and, what is more, we can on occasion combine two particular transformations to get another transformation (just as one can combine two numbers to get another). For example, by combining the two transformations above we get

good → worst, hot → coldest, etc.

So transformations are both connected among themselves and are also another source of connections between the ideas to which they can be applied.

The foregoing offers a brief, and perhaps rather concentrated, glimpse of the richness and variety of the ways in which concepts can be interrelated, and of the resulting structures. The study of the structures themselves is an important part of mathematics, and the study of the ways in which they are built up and function is at the very core of the psychology of learning mathematics.

Now, when a number of suitable components are suitably connected, the resulting combination may have properties which it would have been difficult to predict from a knowledge of the properties of the individual components. How many of us could have predicted from knowledge of the separate properties of transitors, condensers, resistors and the like that, when these are suitably connected, the result would enable us to hear radio broadcasts?

So it is with concepts and conceptual structures. The new function of the electrical structure described above is marked by a new name—transistor radio. Likewise, a conceptual structure has its own name—*schema*. The term includes not only the complex conceptual structures of mathematics but also relatively simple structures which coordinate sensori-motor activity. Here we shall be concerned mainly with abstract conceptual schemas. The previous chapter has shown that these concepts have their origins in sensory experience of, and motor activity towards, the outside world. But they soon become detachable from their origins, and their further development takes place by interaction with other mathematicians and with each other.

Among the new functions which a schema has, beyond the separate properties of its individual concepts, are the following: it integrates existing knowledge, it acts as a tool for future learning and it makes possible understanding.

THE INTEGRATIVE FUNCTION OF A SCHEMA

When we recognize something as an example of a concept we become aware of it at two levels: as itself and as a member of this class. Thus, when we see some particular car, we automatically recognize it as a member of the class of private cars. But this class-concept is linked by our mental schemas with a vast number of other concepts, which are available to help us behave adaptively with respect to the many different situations in which a car can form a part. Suppose the car is for sale. Then all our motoring experience is brought to bear, reviews of its performance may be recalled, questions to be asked (m.p.g.?) present themselves. Suppose that the cost is beyond our present bank balance. Then sources of finance (bank loans, hire purchase) come to mind. Suppose, instead, that the car is on the road, but has broken down. Then instruments of help (such as the A.A., nearest garage, telephone boxes) are recalled.

Most of these schemas have probably already been linked with the car concept in the past. But suppose now that we park on a foreshore and find that our wheels have sunk into the soft sand. This presents a problem, to solve which schemas from other fields of experience must be brought to bear, such as the behaviour of tides, ways of making a firm surface on soft sand. The more other schemas we have available, the better our chance of coping with the unexpected. We shall return to this point later in the chapter.

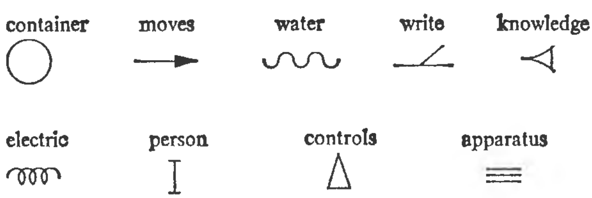
The Schema as a Tool for Further Learning

How information found can be assimilated

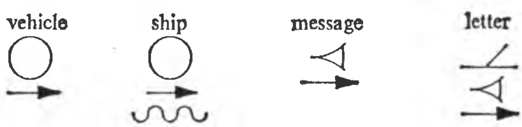
Our existing schemas are also indispensable tools for the acquisition of further knowledge. Almost everything we learn depends on knowing something else already. To learn aircraft designing we must know aerodynamics, which depends on prior knowledge of calculus, which requires knowledge of algebra, which depends on arithmetic. To learn advanced physiology requires biochemistry, which needs a knowledge of elementary 'school' chemistry. These, and all higher learning, depend on the basic schemas of reading, writing and speaking (or, exceptionally, communicating in some other way) our native language.

This principle—the dependency of new learning on the availabiity of a suitable schema—is a generalization of the second principle for conceptual learning, stated in Chapter 1 on page 30. In the generalized form, new features become important which were not so noticeable while we were concentrating on the learning of particular concepts, though using hindsight they can be seen to be latent there. As an introduction to these, it will be useful to look at an experiment[1] whose purpose was to try to isolate the factor of a schema in learning, or more precisely, to find out how much difference the presence or absence of a suitable schema made to the amount of new material which could be learnt.

For the purpose of the experiment, an artificial schema was devised, somewhat resembling a Red Indian sign language. On the first day the subjects learnt the meanings of sixteen basic signs, such as:



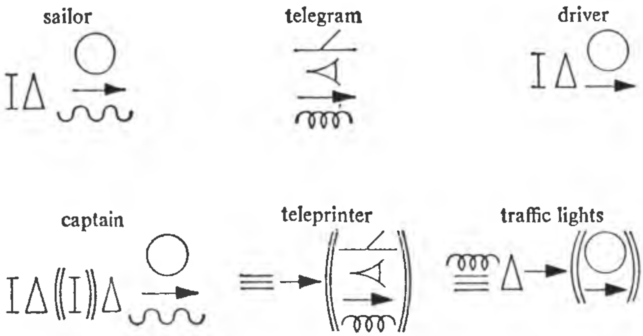
On the second day meanings were assigned to pairs or trios of symbols, such as:



The meanings of these small groups of symbols are related to the meanings of the single symbols, as the reader can verify. On the third and fourth days the groups to be learnt were made progressively larger, the meanings again being related to

[1]This is described fully in Skemp (1962).

These are examples of concept combination, an idea discussed more fully later in the course those of the smaller groups. Here are some examples. (Note that (( )) means plural.)



P.L.M.—3

The final task, on the fourth day, was to learn two pages of symbols, each page containing a hundred symbols in ten groups each having from eight to twelve symbols. On one page each group was given a meaning related to the meanings of the smaller groups, as in the examples given. The other page contained groups which were in fact similarly meaningful to a comparison group, but not to these subjects. The comparison group had learnt the same symbols but with different meanings, and these had been built up into a different schema. So in their final task each group had an appropriate schema for one page and an inappropriate schema for the other page. In other words, what was meaningful (in terms of earlier learning) to one group was non-meaningful to the other, and vice versa.

When the results of schematic and 'rote' learning were compared, the differences were striking.

	% recalled (all subjects)		
	Immediate	*After one day*	*After four weeks*
Schematic	69	69	58
Rote	32	23	8

In this case twice as much was recalled of the schematically learnt as of the rote-learnt material when tested immediately afterwards; and in four weeks the proportion had changed to seven times as much. The schematically learnt material was not only better learnt, but better retained.

Objectively, the two pages of symbols were the same for all the subjects. The only difference was in the mental structures which they had available for the learning task. Clearly, therefore, the schemas which we build up in the course of our early learning of a subject will be crucial to the ease or difficulty with which we can master later topics. When learning schematically—which, in the present context, is to say intelligently—we are not only learning much more efficiently what we are currently engaged in; we are preparing a mental tool for applying the same approach to future learning tasks in that field. Moreover, when subsequently using this tool, we are consolidating the earlier content of the schema. This gives schematic learning a triple advantage over rote memorizing.

There are, however, also certain possible disadvantages to be considered.

The first is that, if a task is considered in isolation, schematic learning may take longer. For example, rules for solving a simple equation (see page 86) can be memorized in much less time than it takes to achieve understanding. So if all one wants to learn is how to do a particular job, memorizing a set of rules may be the quickest way. If, however, one wishes to progress, then the number of rules to be learnt becomes steadily more burdensome until eventually the task becomes excessive. A schema, even more than a concept, greatly reduces cognitive strain. Moreover, in most mathematical schemas, all the main contributory ideas are of very general application in mathematics. Time spent in acquiring them is not only of psychological value (meaning that present and future learning is easier and more lasting) but of mathematical value (meaning that the ideas are also of great importance mathematically). In the present context, good psychology is good mathematics.

The second disadvantage is more far-reaching. Since new experience which fits into an existing schema is so much better remembered, a schema has a highly selective effect on our experience. What does *not* fit into it is largely not learnt at all, and what is learnt temporarily is soon forgotten. So, not only are unsuitable schemas a major handicap to our future learning, but even schemas which have been of real value may cease to become so if new experience is encountered, new ideas need to be acquired, which cannot be fitted in to an existing schema. A schema can be as powerful a hindrance as help if it happens to be an unsuitable one.

This brings us to a consideration of adaptability at a new level. So far a schema has been seen as a major instrument of adaptability, being the most effective organization of existing knowledge both for solving new problems and for acquiring new knowledge (and thereby for solving still more new problems in the future). But its very strength now appears as its potential downfall, in that a strong tendency emerges towards the self-perpetuation of existing schemas. If situations are then encountered for which they are not adequate, this stability of the schemas becomes an obstacle to adaptability. What is then necessary is a change of structure in the schemas: they themselves must adapt. Instead of a stable, growing schema by means of which the individual organizes past experience and *assimilates* new data, *reconstruction* is required before the new situation can be understood. This may be difficult, and if it fails, the new experience can no longer be successfully interpreted and adaptive behaviour breaks down—the individual cannot cope.

An everyday example will illustrate the idea, after which some mathematical examples will be given. Early in life, a child learns to distinguish between compatriots and foreigners. His schema of a foreigner is that of a person who comes from abroad, who speaks English with a different accent from his own, perhaps only with difficulty, whose own language is novel and usually incomprehensible, whose mode of dress and personal appearance are slightly or very different. New individual foreigners and new classes—people from countries he had never heard of—are easily assimilated to this concept, which leads to expansion of his schema. But suppose now that he takes a holiday abroad with his parents and discovers that he himself is described as a foreigner. To him, this is incomprehensible. The local inhabitants are the foreigners; he is British! Before he can comprehend this new experience—assimilate it to his schema—the schema itself has to be restructured. His idea of foreigners has to become that of people in a country which is not their own. Not only does this new concept enable him to understand the new experience and so to behave appropriately; it includes the earlier concept as a special case. This is the best kind of reconstruction.

A schema is of such value to an individual that the resistance to changing it can be great, and circumstances or individuals imposing pressure to change may be experienced as threats—and responded to accordingly. Even if it is less than a threat, reconstruction can be difficult, whereas assimilation of a new experience to an existing schema gives a feeling of mastery and is usually enjoyed.

One of the most basic mathematical schemas which we learn is that of the natural number system—the set of counting numbers together with the operations of addition and multiplication. Having learnt to count to ten, a child rapidly progresses to twenty, and is eager to continue the process. Adding single-figure numbers, with the help of concrete materials, is soon learnt. Extending this to the addition of two-figure numbers requires, first, an understanding of our system of numeration based on place value, but once this has been mastered, addition of three-, four-, five-figure numbers is again a straightforward extension. Multiplication is like repeated addition, long multiplication extends simple multiplication. Throughout, the process is one of expansion.

It is another matter when fractional numbers are encountered. These constitute a new number system, not an extension of one which is known already. The system of numeration is different in itself and has new characteristics: for example, an infinite number of different fractions can be used to represent the same number. Multiplication can no longer be understood in terms of repeated addition. Before fractional numbers can be understood, a major reconstruction of the number schema is required. Some people, indeed, go through life without ever really understanding fractional numbers, and small blame to them. Their teacher probably never understood them either, and the difficulty of this particular reconstruction is such that it would require a child of genius level to achieve it unaided at the age when this task is encountered.

The history of mathematics contains some interesting examples showing the difficulty of reconstruction presented by a new number system. When Pythagoras discovered that the length of the hypotenuse of a right-angled triangle could not always be expressed as a rational number, he swore the members of his school to secrecy about this threat to their existing ways of thinking. In his well-known history of mathematics, Bell (1937) says: 'When negative numbers first appeared in experience, as in debits instead of credits, they, as *numbers,* were held in the same abhorrence as "unnatural" monstrosities as were later the "imaginary" numbers $\sqrt{-1}$, $\sqrt{-2}$, etc.' The Hindu-Arabic system of numerals for the natural numbers also met with great resistance when it was first introduced into Europe in the thirteenth century, and in some places its use was even made illegal. Unspeakable, unnatural, illegal—these are the ways in which the ordinary working tools of present-day mathematics were all characterized by some of the mathematicians who first encountered them. But now that we know the importance of our personal schemas to us, we can begin to understand the defensive nature of these reactions to any new ideas which threaten to overthrow them.

UNDERSTANDING[2]

We are now in a position to say what we mean by understanding. *To understand something means to assimilate it into an appropriate schema.* This explains the subjective nature of understanding and also makes clear that this is not usually an all-or-nothing state. We may achieve a subjective feeling of understanding by assimilation to an inappropriate schema—the Greeks 'understood' thunderstorms by assimilating these noisy affairs to the schema of a large and powerful being, Zeus, getting angry and throwing things. In this case, an appropriate schema involves the idea of an electric spark, so it was not until the eighteenth century that any real understanding of thunderstorms was possible. The first and major step was taken by Benjamin Franklin, who assimilated concepts about thunderstorms to those about electrical discharges. Fuller understanding, however, involves knowledge of ionization processes in the atmosphere—assimilation to a more extensive schema. What happens in a case like this is that the basic schema becomes enlarged and to the original points of assimilation—noise to noise, lightning flash to electric spark—more are added. Better internal organization of a schema may also improve understanding, and clearly there is no stage at which this process is complete. One obstacle to the further increase of understanding is the belief that one already understands fully.

We can also see that the deep-rooted conviction mentioned earlier, that it matters whether or not we understand something, is well-founded. For this subjective feeling that we understand something, open to error though it may be, is in general a sign that we are therefore now able to behave appropriately in a new class of situations.

[2] I mean here relational understanding. See Chapter 12.

5

Symbols

In previous chapters we have considered the formation of concepts, the function of schemas (conceptual structures) in integrating existing knowledge and assimilating new knowledge, and the additional power which comes from the ability to reflect on one's schemas. In each of these processes an essential part is played by symbols, which have other functions as well. It is now time to consider these in detail.

Among the functions of symbols, we can distinguish:

(i) Communication
(ii) Recording knowledge
(iii) The communication of new concepts
(iv) Making multiple classification straightforward
(v) Explanations
(vi) Making possible reflective activity
(vii) Helping to show structure
(viii) Making routine manipulations automatic
(ix) Recovering information and understanding
(x) Creative mental activity

Most of these are related, particularly to the first. Recording knowledge is communicating with the reader, explanation is a special kind of communication, reflecting is communicating within oneself; and other connections will also be apparent. The headings are therefore intended for convenience only, as starting points for the discussions which follow, not as partitions.

COMMUNICATION

A concept is purely mental object—inaudible and invisible. Since we have no way of observing directly the contents of someone else's mind, nor of allowing others access to one's own, have to use means which are either audible or visible—spoken words or other sounds, written words or other marks on paper (notations). A symbol is a sound, or something visible, mentally connected to an idea. This idea is the *meaning* of the symbol. Without an idea attached, a symbol is empty, meaningless.

Provided that a symbol is connected to the same concept in the minds of two people, then by uttering[1] this symbol, one can evoke the concept from the other's memory into their consciousness—can cause them to 'think of' this concept in the present. This proviso is, however, no small one. Once the connection is established, its meaning is projected on to the symbol, and the two are perceived as a unity. So it is hard to realize that what is meaningful to oneself may not be meaningful to the hearer—a difficulty experienced by many when speaking to foreigners—or that the same meaning is not being attached, for example, to the word 'braces', which may mean to someone British a device for holding up one's trousers, but to an American a pair of set brackets { }. We may think that we are communicating when we are not, and, indeed, it is impossible to know for certain whether we are, and, if so, to what degree. For the reason given above, we usually take it for granted, but the communication links are so precarious, and so inaccessible to study, that we would do better to be surprised that we can communicate our ideas to each other at all. After all, it has taken millions of years of evolution to produce an animal which can do so to any marked extent.

Let us take as a starting point (a) that a symbol and the associated concept are two different things; (b) that this distinction is non-trivial, being that between an object and the name of that object. If an object is called by another name, we do not change the object itself, and this is still true for an object of thought—in the present context, a mathematical idea. For example,

'five,' '*cinq*,' '5,' '*V*,' '101'

all refer to the same number in different notations. We do not call five an English number and *cinq* a French number, nor should we call 5 an Arabic number and V a Roman number. But we still read, all too often, instructions to pupils like "Turn the binary number 101 into a decimal number.' The whole object is, of

---

[1]This will be used as a convenient condensation for speaking, writing, drawing, projecting on a screen, etc.

course, *not* to change the number itself in the process of representing it in a different way. In translating from French into English, we try to keep the meaning the same while changing the words. In converting pounds to dollars we try to keep the value in goods or services the same while representing this value by different tokens (coins, notes) or symbols (figures on a cheque or bank transfer).

The term 'binary number' also implies that being binary is a property which a number can have or not, like being even, prime, an integer, etc. But binary *numerals* can be used to represent any kind of number at all, odd or even, prime or factorizable, natural number, integer, rational or real number. One of the first requirements of communicating an idea is to be clear about it oneself. Those who talk or write about 'binary numbers' and 'decimal numbers' are not.

Usually, when uttering a symbol, we want to call to the attention of the receiver the idea attached to the symbol rather than the symbol itself. If it is the symbol we are referring to, we can show this by quotation marks. (More symbols! They are inescapable.) Example:

'5' and 'V' are both symbols for (the number) five.

A symbol for a number is called a 'numeral', and a system of numeration is a system for writing as many different numbers as we like with a relatively small number of digits (single numerals like 0, 1, 2, 3, 4 . . . 9). The decimal system uses ten digits, the binary system uses two. If it is not clear from the context which system is in use, this can be shown simply and clearly by a suffix. The sign = ('is equal to') means that we are referring to the same concept, (usually) by different symbols. So, for example,

$5_{ten} = 101_{two}$ (since 101 in binary means the same as 5 in decimal notation.)

Similarly $8_{ten} = 10_{eight} = 1000_{two}$ etc.

But '8_{ten}' $\neq$ '10_{eight}.' The numbers are equal, the numerals are different

Excessive precision in the use of language[2] is rightly regarded as pedantry. So it is a fair question at this stage to ask whether this label is applicable to the preceding discussion. Does it really matter, for example, which of these we say or write:

'Write the binary number 11010 as a decimal number' or
'Write 11010_{two} in decimal notation'?

An easy defence would be to claim that it is part of the duty of a mathematician to be as accurate as possible all the time. But this, though plausible, is not valid. It would, for example, imply that we should never use convenient but

[2]A symbol system; for example, the English language, the language of mathematics.

loose phrases such as 'as small as we like.' Part of the aim of mathematics is, by abstraction and the omission of irrelevancies, to enable us 'to see the wood for the trees,' and this will not be achieved by adding, instead, a mass of mathematical detail in the name of accuracy.

The kind of accuracy with which we are at present concerned is accuracy of communication, with trying to get as near as we can to the impossibility of producing the same idea in the mind of the receivers as of the communicators or calling it to their attention.

Now, we can distinguish three categories of hearer or reader. First, those who don't yet know what we are talking about, but want to. For these, we should choose our symbols with the greatest possible care and use them as accurately as we can, with the aim of communicating nothing but the truth, though not yet necessarily the whole truth. Concepts are built up by degrees. The first approximation is bound to be incomplete, and perhaps to need tidying up in detail, but there should not be anything of importance to un-learn. It is also worth bearing in mind that, to an intelligent learner, a brief but inaccurate statement may well be more confusing than a somewhat lengthier, but accurate, statement.

The second category comprises those who do know what we are talking about, as a general background within which we are trying to communicate some particular aspect. If they are willing to 'go along' with us, we can take much for granted, save time and concentrate on essentials. An old and wise teacher of mine often used, in the context of limits and convergence, phrases like 'As near as dammit to . . .' We both knew what he meant, and both could, if necessary, have re-phrased it in rigorous terms. So, for the task in hand, the idea was communicated with complete accuracy by this short and expressive phrase.

The third category of hearer or reader consists of those who do know what we are talking about but want to fault it. A non-mathematical example of this activity is to be found every time a new tax is made law. The finance minister says 'I want a tax on . . .' As soon as this becomes law, an army of expert accountants will go to work on behalf of their clients to see how this tax can legally be avoided or reduced. So, before the bill goes through, the parliamentary draughtsmen have to try to stop all loopholes in advance. The result is to make it almost unintelligible.

Similarly in mathematics, rigour and ease of understanding do not go together. The art of communication is, first, to convey meaning. Afterwards, the new ideas can be subjected to the stress of analysis, and greater precision introduced where weaknesses are found. The difference is that, once a schema is well established, this critical attack serves a useful purpose, that of stimulating more careful formulation and greater reflective awareness, and the strengthening of the schema without loss of integration of 'the overall picture.' This criticism may come from another person or it may come from a 'devil's advocate' within oneself. This seems to be another function of the reflective system—to take 'an outside view' of an argument or other intended communication, and by self-criticism anticipate external criticism.

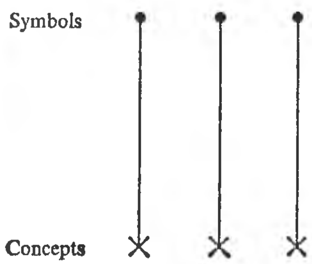
RECORDING KNOWLEDGE

Ideas are not only invisible and inaudible, they are perishable. When we die, our knowledge dies with us, unless we have communicated or recorded it. One of the most moving episodes in the history of mathematics is that of the young Galois (1811–32) sitting up all night, writing against time to commit to paper his theory of groups, before his tragic and wasteful death by duel at the age of twenty.

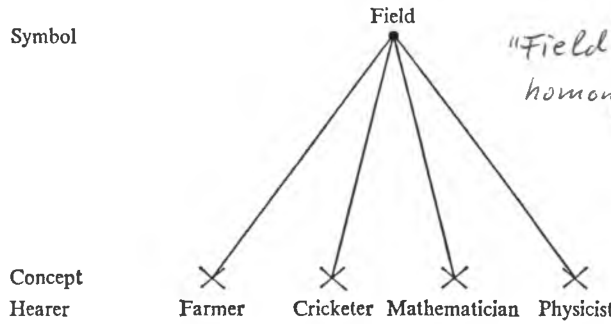
Recording is a special case of communicating, since it is normally done with the intention that these records will, in the near or distant future, be seen by others. So all the previous section applies. Whereas the spoken communication usually (though not always) takes place in circumstances which allow questions and explanations to be given, written or printed symbols have to convey all the required meaning, without a second chance on either side. So the communicators have to take more trouble to try to ensure this. There is, however, the advantage that the receivers have a permanent record for revision and the checking of earlier points. They can also go at a speed to suit their own rate of assimilation.

As has been discussed in Chapter 1, the conceptual structure of mathematics is something far beyond that which anyone could construct, unaided, in a lifetime. Limited areas have taken years of work by some of the world's most gifted individuals. It is the storage of the accumulated knowledge of previous generations by written and printed symbol systems (and recently by other techniques such as recording tape, cinematography, microfilm), together with the auxiliary explanations of live teachers, that make it possible for some of each new generation to learn in years ideas which took centuries of collective effort to form for the first time, in each case, synthesizing them anew, and in some cases building new knowledge and adding this to the store.

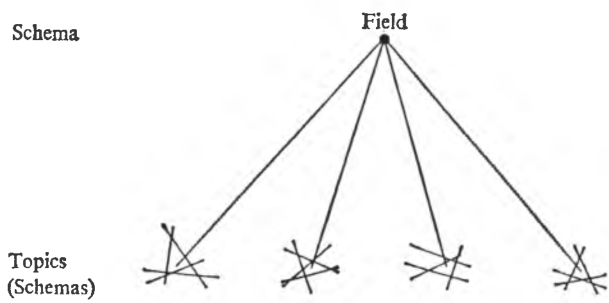
concept - symbol (or term) relationship, see Ch.12 of the course text



One of the first requirements for the avoidance of ambiguity which one would expect to be observed is that each symbol is associated with one concept, and vice versa. This arrangement is, however, seldom found in practice, even in a single language. Mathematicians seem to be particularly lazy about inventing new symbols, relying largely on the capital and lower-case letters of the Roman alphabet, the Greek alphabet, punctuation marks and the like, each of which does multiple duty. So a single symbol may well stand for a variety of concepts.



The arrangement just shown might be expected to lead to confusion, since the word 'field' will evoke different concepts in the minds of each of the individuals named above. Or, if we are addressing someone with interests in all these topics, then we cannot be sure which concept will be evoked by the word 'field' in isolation. But, of course, the word is seldom used in isolation. Ordinarily the



hearer knows which topic is under discussion, and only ideas within this topic are accepted as possible meanings for the word. If not, then the speaker or writer uses one or more other symbols to evoke the relevant schema as a whole. This established a 'set'—a state of mind in which concepts belonging to this particular schema are more easily evoked. Symbols used in this way, to determine the schema within which a particular symbol takes its meaning, are called its *context*.

From this, three simple rules can be formulated for conveying the desired meaning when one symbol corresponds to many concepts.

1. Be sure that the schema in use is known to the hearer or reader.
2. Within this schema let each symbol represent only one idea.
3. Do not change schemas without the knowledge of the hearer or reader.

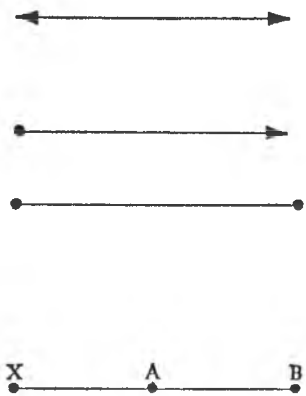
It is permissible (though whether any advantage is gained is another question) to use the same symbol in different contexts with different meanings. But in the same context a symbol must have just one meaning. So we can write $AA' = I$ in the context of matrices, and $AA' = BB'$ in the context of points and lines, without confusion. But if we write $(x + a)^2 = x^2 + 2ax + a^2$ the x and the a must keep the same meaning throughout, because they are in a single context.

These rules seem straightforward and obvious, but they are not always observed, with the result that the learner is confused. Here is an example.

Children first learn the meaning of multiplying in the context of natural numbers, which refer to sets of discrete, countable objects. So the operations 3×4 corresponds to combining four sets, each of three objects, and counting the objects in the resulting set.[3] They use the sign '×' with this meaning for several years, and it is the only meaning they know. We then change to a new number system, say, fractional numbers or integers, in which the sign (or word) has a different meaning. But we do not tell the children that we have changed the context and have generalized the meaning of '×' to suit the new context. So they no longer fully understand what they are doing.

If the new context was very different from the old, children would probably discover what was happening unaided. But the contexts are sufficiently alike to make it hard for them to do so. One way of indicating the change is already in use in advanced texts. The symbol '⊗' (and also '⊕') is used in the new context, to show that these operations are like the others but that we must not expect them to be quite the same. The readers of these texts probably come into the third of the categories outlined on page 49, those who will be quick to notice any inaccuracy. But those for whom accuracy of communication is most necessary are those in the first category, those who do not yet know what we are talking about but want to. When these pass on into category two, we may conveniently revert to the symbols '+' and '×', since they are now able to assign the appropriate meanings according to context.

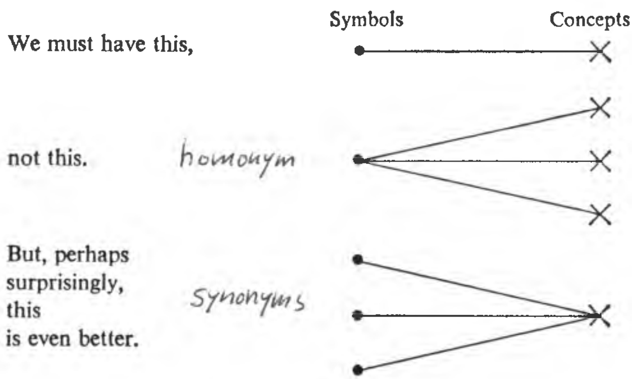
The word 'line' is commonly used with at least three different meanings: (a) a line indefinite length, extending indefinitely in both directions; (b) one which starts at a given point and extends indefinitely in one direction from it; and one which is of finite length, bounded by two points. These three meanings may conveniently be distinguished by the terms 'line,' 'ray' and 'line segment.' So the point X is on the line AB (or BA), and also on the ray BA; but it is not on the ray AB, nor on the line segment AB. If AB represents a railway line, X our destination and A our starting point, the distinction is hardly trivial!



[3]That is, assuming that we read '3 × 4' as 'three multiplied by four.' It is also read by some as 'three times four,' which corresponds to combining three sets, each of four objects. We should be more surprised than we are that both of these give the same result.

The mathematically experienced reader should have no difficulty in finding other examples of the ambiguous use of symbols. Some suggestions: what is meant by '$AB = 3$ cm?' What is meant by 'the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ etc.?' And in the context of groups, are the terms 'identity element' and 'neutral element' synonymous?

So far, the emphasis of this section has been that, in a given context (which may be explicit or implicit), one symbol should represent only one concept. What matters is the meaning (the associated concept), and provided that each



symbol conveys only one meaning, it is often an advantage to have a choice. If A uses a term (for example, 'cuboid') which is unfamiliar to B, they can try again with another (say, 'rectangular block'). The choice of symbol also enables us to classify the same idea in different ways, a use which will be discussed further in section (iv) of this chapter; and, related to this, it can help us to emphasize that aspect of a complex idea which is most relevant to particular circumstances. For example, *function* is a concept with widespread applications, and in Chapter 13 we shall see that there are no less than six useful ways of representing a given function.

Other advantages of using several different symbols for the same concept will be mentioned later in this chapter. If we do this, however, an obvious precaution is necessary to ensure that the reader knows that we are in fact talking about the same thing, though using different names; and this becomes more important when recording mathematics, as distinct from communicating face-to-face, since the reader cannot ask. This is the meaning of the symbol '=,' that the symbols on each side of the sign of equality refer to the same object.

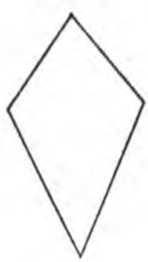
THE COMMUNICATION OF NEW CONCEPTS

It will be recalled that in Chapter 2 the point was made that new concepts of a higher order than those which the learner already has can only be communicated by arranging for the learner to group together mentally a suitable set of examples.

If the new concept is a primary concept, for example, red, it is possible to do this without the use of symbols, simply by pointing. The words 'This is a . . .' simply help to draw attention; they are verbal pointers. 'Red tie,' 'red book,' 'red pencil,' 'red light,' however, express simultaneously the variability of the examples and the constancy of the concept. Intuitively the learner associates the invariant property with the invariant word, and so learns the name for the concept while it is being formed.

If the concept is a secondary concept, as are all mathematical concepts, then the only way of bringing together a suitable set of examples in the learner's mind is to bring together the corresponding words. 'Red, blue, green, yellow—these are all colours.' By manipulating the words we manipulate the minds of the learners—normally, with their consent. (If they feel otherwise, there will naturally be resistance to learning: see Chapter 7.) In this way learners may be helped to see something in common between examples which, separately encountered over an interval of time, would have remained isolated in their minds. It took Newton to perceive for the first time something in common between the fall of an apple and the motion of the planets round the sun; but when he brings these ideas together for us, we too can form the concept of gravitation.

Another way of communicating new concepts is by relating together classes already known to the hearer. 'What is a Sinhalese?' 'An inhabitant of Sri Lanka.' 'What is a kite?' (In the context of geometry.) 'A quadrilateral with two pairs of adjacent sides equal.' 'What is a variable?' 'An unspecified member of a given set.' If the hearer already has the class concepts mentioned, this implies that examples of these are known, so it should also be possible to supply examples of these new concepts. Indeed, this is often the first response, partly to confirm that the concept has been understood. (Sketching rapidly in response to the second definition: 'Like this?')



But the response also seems to satisfy a deeper need. Somehow, a concept acquired in the way just described seems incomplete until it has some examples. A tentative explanation of this is that a concept confers the ability to class together an appropriate set of examples, and it is generally observable that the acquisition of a new ability often seems to carry with it a need to exercise it. (Give your small son a kit of tools for his birthday and observe the result.)

The examples of the new concept thus supplied need not be from past experience. One can imagine a Sinhalese without ever having met one; one can imagine a 100-sided regular polygon without having seen one and without having to draw one. Indeed, a fruitful and exciting method of mathematical generalization is to invent a new class, and then try to find some members of it. Example: suppose that we already have the concepts square root and negative number, and combine these to form a new concept—the square root of a negative number. The search for examples of this new class, and the investigation of their properties, leads to the construction of a new set of ideas which, though termed 'imaginary' numbers, are nevertheless of great practical use in physics: for example, in the theory of alternating current and oscillatory circuits.

MAKING MULTIPLE CLASSIFICATION STRAIGHTFORWARD

polyhierarchy (Ch. 14 of course text)

A single object may be classified in many different ways, and, by using different names for it (which we have already seen to be permissible), we can indicate what particular classification is currently in use. The same man may be called 'Mr John Brown,' 'Sir,' 'The right honourable gentleman,' 'Uncle Jack,' 'Daddy,' or 'John.' The same angle may be classified as the angle vertically opposite to . . . or as the third angle of triangle . . . The same number may be regarded as the square of 8, the cube of 4 or the square of 10 minus the square of 6, may be symbolized by 8^2, 4^3, $10^2 - 6^2$. By our choice of symbol, we are enabled to concentrate our attention on different properties of the same object.

As already noted, we show that we are still (often in spite of appearances) referring to the same object[4] by the symbol '=,' and, by renaming according to already established routines, we can find properties which were at first not apparent.

Example: $4x^2 - 12xy + 9y^2$, where x and y are both numerical variables (unspecified numbers). We know that this collection of symbols represents some number. But by writing

$$4x^2 - 12xy + 9y^2 = (2x - 3y)^2$$

we know something new—that it represents a *positive* number.

Though the principle is a simple one, its consequences are far-reaching. Once we have appropriately classified something, we are a long way towards knowing how to deal with it. (This polite caller—is he a salesman, a public-opinion surveyor or a plain-clothes detective? Our response is cautious until we know which.) 'Appropriately' means in a way (or ways) which helps us to solve the problem in hand; and so the more ways in which we can classify, the greater the variety of problems which we can solve. And the more symbols we can attach to the same ~~object~~ concept, the more ways we can classify

[4]Reminder: this, in the present context, usually means an object of thought.